Robust Standard Errors in Small Samples

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Description

This package implements small-sample degrees of freedom adjustments to robust and cluster-robust standard errors in linear regression, as discussed in Imbens and Kolesár [2016]. The implementation can handle models with fixed effects, and cases with a large number of observations or clusters [1].

```
library(dfadjust)
```

To give some examples, let us construct an artificial dataset with 11 clusters

```
set.seed(7)
d1 <- data.frame(y = rnorm(1000), x1 = c(rep(1, 3), rep(0,
    997)), x2 = c(rep(1, 150), rep(0, 850)), x3 = rnorm(1000),
    cl = as.factor(c(rep(1:10, each = 50), rep(11, 500))))
```

Let us first run a regression of `y` on `x1`. This is a case in which, in spite of moderate data size, the effective number of observations is small since there are only three treated units:

```
r1 <- lm(y ~ x1, data = d1)
## No clustering
dfadjustSE(r1)
#>
#> Coefficients:
#>             Estimate HC1 se HC2 se Adj. se     df p-value
#> (Intercept)  0.00266 0.0311  0.031  0.0311 996.00   0.932
#> x1           0.12940 0.8892  1.088  2.3743   2.01   0.916
```

We can see that the usual robust standard errors (`HC1 se`) are much smaller than the effective

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[1] We thank Ulrich Müller for suggesting to us the lemma below.

standard errors (`Adj. se`), which are computed by taking the `HC2` standard errors and applying a degrees of freedom adjustment.

Now consider a cluster-robust regression of `y` on `x2`. There are only 3 treated clusters, so the effective number of observations is again small:

```
r1 <- lm(y ~ x2, data = d1)
# Default Imbens-Kolesár method
dfadjustSE(r1, clustervar = d1$cl)
#>
#> Coefficients:
#>             Estimate HC1 se HC2 se Adj. se    df p-value
#> (Intercept)  -0.0236 0.0135 0.0169  0.0222 4.94  0.2215
#> x2            0.1778 0.0530 0.0621  0.1157 2.43  0.0826
# Bell-McCaffrey method
dfadjustSE(r1, clustervar = d1$cl, IK = FALSE)
#>
#> Coefficients:
#>             Estimate HC1 se HC2 se Adj. se    df p-value
#> (Intercept)  -0.0236 0.0135 0.0169  0.0316 2.42  0.2766
#> x2            0.1778 0.0530 0.0621  0.1076 2.70  0.0731
```

Now, let us run a regression of `y` on `x3`, with fixed effects. Since we're only interested in `x3`, we specify that we only want inference on the second element:

```
r1 <- lm(y ~ x3 + cl, data = d1)
dfadjustSE(r1, clustervar = d1$cl, ell = c(0, 1, rep(0,
    r1$rank - 2)))
#>
#> Coefficients:
#>          Estimate HC1 se HC2 se Adj. se    df p-value
#> Estimate   0.0261 0.0463 0.0595  0.0928 3.23   0.688
dfadjustSE(r1, clustervar = d1$cl, ell = c(0, 1, rep(0,
    r1$rank - 2)), IK = FALSE)
#>
#> Coefficients:
#>          Estimate HC1 se HC2 se Adj. se    df p-value
#> Estimate   0.0261 0.0463 0.0595  0.0928 3.23   0.688
```

Finally, an example in which the clusters are large. We have 500,000 observations:

```
d2 <- do.call("rbind", replicate(500, d1, simplify = FALSE))
d2$y <- rnorm(length(d2$y))
r2 <- lm(y ~ x2, data = d2)
summary(r2)
#>
#> Call:
#> lm(formula = y ~ x2, data = d2)
#>
#> Residuals:
```

```
#>     Min     1Q Median     3Q     Max
#> -5.073 -0.675  0.000  0.675  4.789
#>
#> Coefficients:
#>             Estimate Std. Error t value Pr(>|t|)
#> (Intercept) -0.000991   0.001535   -0.65     0.52
#> x2          -0.003590   0.003963   -0.91     0.37
#>
#> Residual standard error: 1 on 499998 degrees of freedom
#> Multiple R-squared:  1.64e-06,   Adjusted R-squared:  -3.59e-07
#> F-statistic: 0.821 on 1 and 5e+05 DF,  p-value: 0.365
# Default Imbens-Kolesár method
dfadjustSE(r2, clustervar = d2$cl)
#>
#> Coefficients:
#>             Estimate  HC1 se  HC2 se Adj. se    df p-value
#> (Intercept) -0.000991 0.00133 0.00168 0.00294 2.66   0.603
#> x2          -0.003590 0.00483 0.00568 0.00997 2.65   0.578
# Bell-McCaffrey method
dfadjustSE(r2, clustervar = d2$cl, IK = FALSE)
#>
#> Coefficients:
#>             Estimate  HC1 se  HC2 se Adj. se    df p-value
#> (Intercept) -0.000991 0.00133 0.00168 0.00315 2.42   0.607
#> x2          -0.003590 0.00483 0.00568 0.00984 2.70   0.577
```

Methods

This section describes the implementation of the Imbens and Kolesár [2016] and Bell and McCaffrey [2002] degrees of freedom adjustments.

There are S clusters, and we observe n_s observations in cluster s, for a total of $n = \sum_{s=1}^{S} n_s$ observations. We handle the case with independent observations by letting each observation be in its own cluster, with $S = n$. Consider the linear regression of a scalar outcome Y_i onto a p-vector of regressors X_i,

$$Y_i = X_i'\beta + u_i, \qquad E[u_i \mid X_i] = 0.$$

We're interested in inference on $\ell'\beta$ for some fixed vector $\ell \in \mathbb{R}^p$. Let X, u, and Y denote the design matrix, and error and outcome vectors, respectively. For any $n \times k$ matrix M, let M_s denote the $n_s \times k$ block corresponding to cluster s, so that, for instance, Y_s corresponds to the outcome vector in cluster s. For a positive semi-definite matrix M, let $M^{1/2}$ be a matrix satisfying $M^{1/2'}M^{1/2} = M$, such as its symmetric square root or its Cholesky decomposition.

Assume that

$$E[u_s u_s' \mid X] = \Omega_s, \quad \text{and} \quad E[u_s u_t' \mid X] = 0 \quad \text{if } s \neq t.$$

Denote the conditional variance matrix of u by Ω, so that Ω_s is the block of Ω corresponding to cluster s. We estimate $\ell'\beta$ using OLS. In R, the OLS estimator is computed via a QR decomposition,

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$X = QR$, where $Q'Q = I$ and R is upper-triangular, so we can write the estimator as

$$\ell'\hat{\beta} = \ell' \left(\sum_s X_s'X_s\right)^{-1} \sum_s X_sY_s = \tilde{\ell}' \sum_s Q_s'Y_s, \qquad \tilde{\ell} = R^{-1'}\ell.$$

It has variance

$$V := \text{var}(\ell'\hat{\beta} \mid X) = \ell' \left(X'X\right)^{-1} \sum_s X_s'\Omega_s X_s \left(X'X\right)^{-1} \ell = \tilde{\ell}' \sum_s Q_s'\Omega_s Q_s \tilde{\ell}.$$

Variance estimate

We estimate V using a variance estimator that generalizes the HC2 variance estimator to clustering. Relative to the LZ2 estimator described in Imbens and Kolesár [2016], we use a slight modification that allows for fixed effects:

$$\hat{V} = \ell'(X'X)^{-1} \sum_s X_s'A_s\hat{u}_s\hat{u}_s'A_s'X_s(X'X)^{-1}\ell = \ell'R^{-1}\sum_s Q_s'A_s\hat{u}_s\hat{u}_s'A_s'Q_sR'^{-1}\ell = \sum_{s=1}^S (\hat{u}_s'a_s)^2,$$

where
$$\hat{u}_s := Y_s - X_s\hat{\beta} = u_s - Q_sQ'u, \qquad a_s = A_s'Q_s\tilde{\ell},$$

and the matrix A_s is given by the symmetric square root of the inverse of $I - Q_sQ_s'$, or else its pseudo-inverse if it is singular, as is the case, for example, if X contains fixed effects. We do not need to insist on $I - Q_sQ_s'$ to be invertible, since, using the identity

$$\hat{V} = u \sum_s (I - QQ')_s'a_sa_s'(I - QQ')_s u,$$

one can verify by simple algebra that a sufficient condition for $\hat{V}$ to be unbiased under homoskedasticity is that $Q_s'A_s(I - Q_sQ_s')A_sQ_s = Q_s'Q_s$ (see, for example, Pustejovsky and Tipton [2018], for details).

If the observations are independent, the vector of leverages $(Q_1'Q_1, \ldots, Q_n'Q_n)$ can be computed directly using the `stats::hatvalues` function. In this case, use this function to compute $A_i = 1/\sqrt{1 - Q_i'Q_i}$ directly, and we then compute $a_i = A_iQ_i'\tilde{\ell}$ using vector operations. For the case with clustering, computing the spectral decomposition of $I - Q_sQ_s'$ can be expensive or even infeasible if the cluster size n_s is large. We therefore use the following result, suggested to us by Ulrich Müller, which allows us to compute a_s by computing a spectral decomposition of a $p \times p$ matrix.

- Let $Q_s'Q_s = \sum_{i=1}^p \lambda_{is}r_{is}r_{is}'$ be the spectral decomposition of $Q_s'Q_s$. Then $A_s = \sum_{i: \lambda_i \neq 1}(1 - \lambda_i)^{-1/2}Q_sr_{is}r_{is}'Q_s'$, satisfies $A_s(I - Q_sQ_s')A_s = I$.

  This follows from the fact that $I - Q_sQ_s'$ has eigenvalues $1 - \lambda_{is}$ and eigenvectors Q_sr_{is}, and hence its pseudoinverse is $\sum_{i: \lambda_i \neq 1}(1 - \lambda_i)^{-1}Q_sr_{is}r_{is}'Q_s'$.

Using the lemma, we can compute a_s efficiently as:

$$a_s = \sum_{i: \lambda_i \neq 1}(1 - \lambda_i)^{-1/2}Q_sr_{is}r_{is}'Q_s'Q_s\tilde{\ell} = Q_sD_s\tilde{\ell}, \qquad D_s = \sum_{i: \lambda_i \neq 1}\lambda_i(1 - \lambda_i)^{-1/2}r_{is}r_{is}'.$$

Degrees of freedom correction

Let G be an $n \times S$ matrix with columns $(I - QQ')'_s a_s$. Then the Bell and McCaffrey [2002] adjustment sets the degrees of freedom to

$$f_{\text{BM}} = \frac{\text{tr}(G'G)^2}{\text{tr}((G'G)^2)}.$$

Since $(G'G)_{st} = a'_s(I - QQ')_s(I - QQ)'_t a_t = a_s(\mathbb{1}\{s = t\} - Q_s Q'_t)a_t$, the matrix $G'G$ can be efficiently computed as

$$G'G = \text{diag}(a'_s a_s) - BB' \qquad B_{sk} = a'_s Q_{sk}.$$

Note that B is an $S \times p$ matrix, so that computing the degrees of freedom adjustment only involves $p \times p$ matrices:

$$f_{\text{BM}} = \frac{(\sum_s a'_s a_s - \sum_{s,k} B^2_{sk})^2}{\sum_s (a'_s a_s)^2 - 2\sum_{s,k}(a'_s a_s)B^2_{sk} + \sum_{s,t}(B'_s B_t)^2}.$$

If the observations are independent, we compute B directly as `B <- a*Q`, and since a_i is a scalar, we have

$$f_{\text{BM}} = \frac{(\sum_i a^2_i - \sum_{sk} B^2_{sk})^2}{\sum_i a^4_i - 2\sum_i a^2_i B'_i B_i + \sum_{i,j}(B'_i B_j)^2}.$$

The Imbens and Kolesár [2016] degrees of freedom adjustment instead sets

$$f_{IK} = \frac{\text{tr}(G'\hat{\Omega}G)^2}{\text{tr}((G'\hat{\Omega}G)^2)},$$

where $\hat{\Omega}$ is an estimate of the Moulton [1986] model of the covariance matrix, under which $\Omega_s = \sigma^2_\epsilon I_{n_s} + \rho \iota_{n_s} \iota'_{n_s}$. Using simple algebra, one can show that in this case,

$$G'\Omega G = \sigma^2_\epsilon \text{diag}(a'_s a_s) - \sigma^2_\epsilon BB' + \rho(D - BF')(D - BF')',$$

where

$$F_{sk} = \iota'_{n_s} Q_{sk}, \qquad D = \text{diag}(a'_s \iota_{n_s})$$

which can again be computed even if the clusters are large. The estimate $\hat{\Omega}$ replaces σ^2_ϵ and ρ with analog estimates.

References

Robert M. Bell and Daniel F. McCaffrey. Bias reduction in standard errors for linear regression with multi-stage samples. *Survey Methodology*, 28(2):169–181, December 2002.

Guido W. Imbens and Michal Kolesár. Robust standard errors in small samples: Some practical advice. *Review of Economics and Statistics*, 98(4):701–712, October 2016. doi: 10.1162/REST_a_005 52.

Brent R. Moulton. Random group effects and the precision of regression estimates. *Journal of Econometrics*, 32(3):385–397, August 1986. doi: 10.1016/0304-4076(86)90021-7.

James E. Pustejovsky and Elizabeth Tipton. Small-sample methods for cluster-robust variance estimation and hypothesis testing in fixed effects models. *Journal of Business & Economic Statistics*, 36(4):672–683, October 2018. doi: 10.1080/07350015.2016.1247004.